There is more to ensemble models than model weights

Michael A. Spence

30th November 2022

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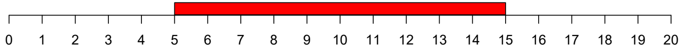

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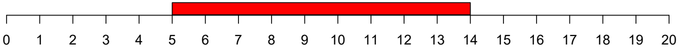

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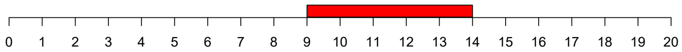

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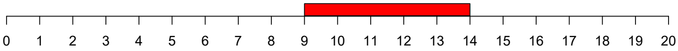

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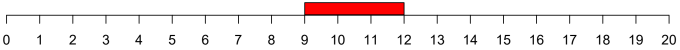

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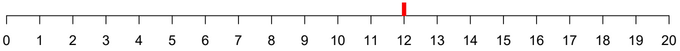

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and after Jim's information

$$p(y|x_{\text{Kelly}}, x_{\text{Jim}}) \sim \text{unif}\{5, 14\}$$

and so on...

Average (possibly weighted)

The estimate of the truth $\tilde{y}$ is

$$\tilde{y} = \sum_{i=1}^{m} w_i x_i,$$

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where $\sum_{i=1}^{m} w_i = 1$. If

$$E(x_i) = y$$

then

$$\lim_{m \to \infty} \tilde{y} = y.$$

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Why would this be true? Model can be

- fitted to the same data

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Average (possibly weighted)

Why would this be true? Model can be

- fitted to the same data
- built with same knowledge
- have similar processes
- fitted by similar or sometimes the same people

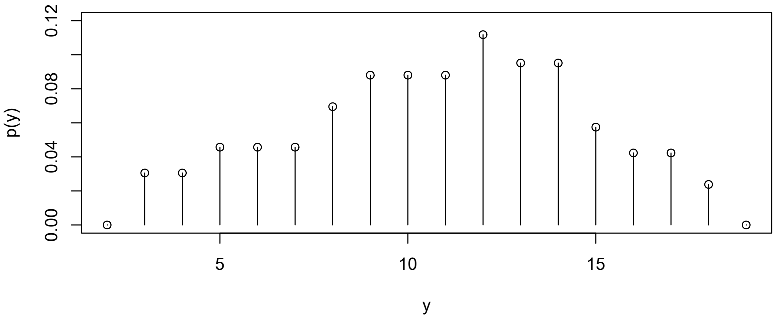
Weighted model

One of the models is right (e.g. Bayesian model averaging), implies

$$p(y) = \sum_{i=1}^{m} w_i p(y|x_i),$$

where $\sum_{i=1}^{m} w_i = 1$.

Weighted model – example equally weighted



Weighted model – example equally weighted – before Angela

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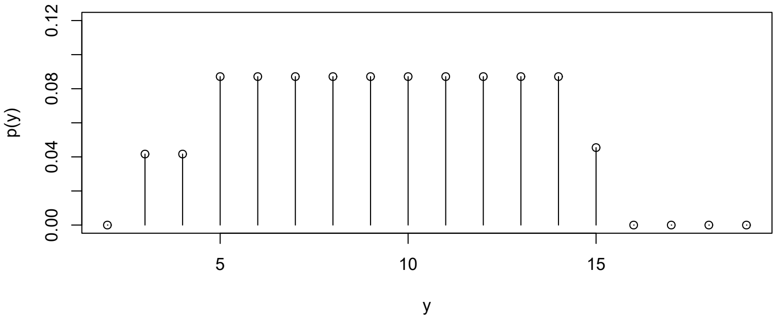
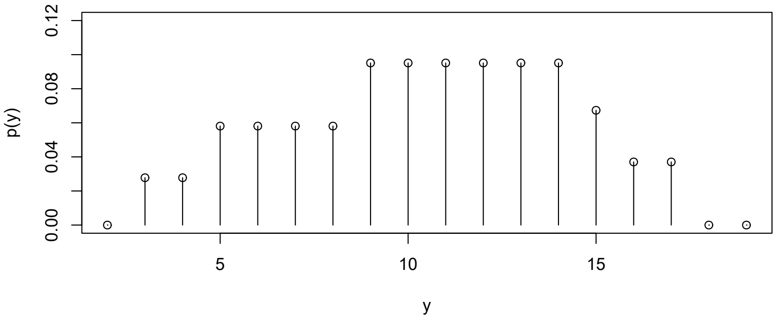


Weighted model – example equally weighted – after Angela



Weighted models – adding an $m + 1$ model

With m models we asked,

One of these m models is the truth, which one?

Weighted models – adding an $m+1$ model

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and then when we have $m+1$ models we ask,

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Adding to our knowledge

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&= \frac{p(y)p(x_1|y)p(x_2|y,x_1)\ldots p(x_m|y,x_1,\ldots,x_{m-1})}{p(x_1)p(x_2|x_1)\ldots p(x_m|x_1,\ldots,x_{m-1})}.
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What if we added an $m+1$ model? Then,

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Hence we need to know

$$p(x_{m+1}|y,x_1,\dots,x_m).$$

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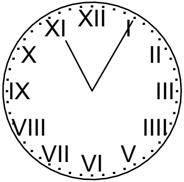
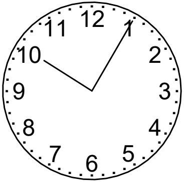
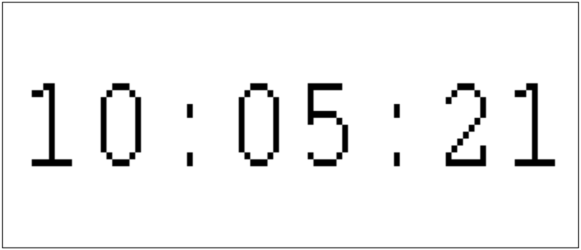
where δ_i is the discrepancy.

Broken clocks







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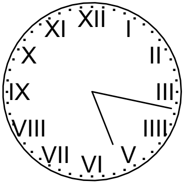

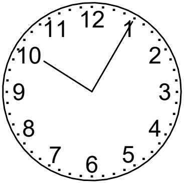

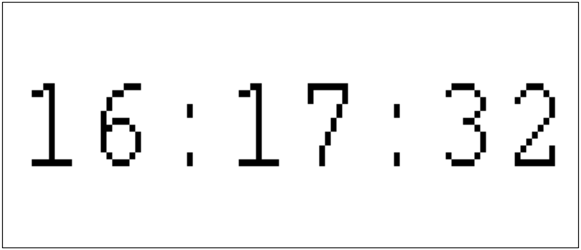

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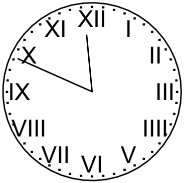

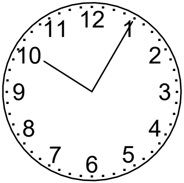

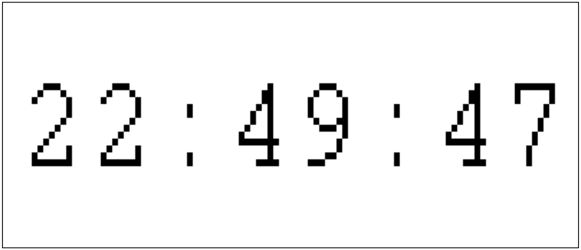

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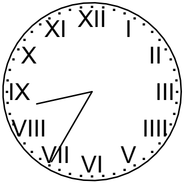

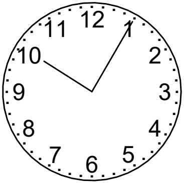

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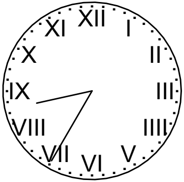

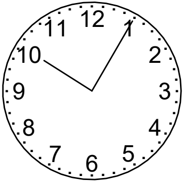

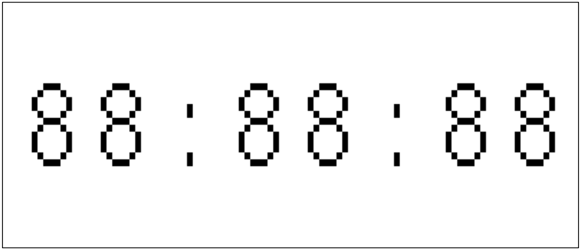

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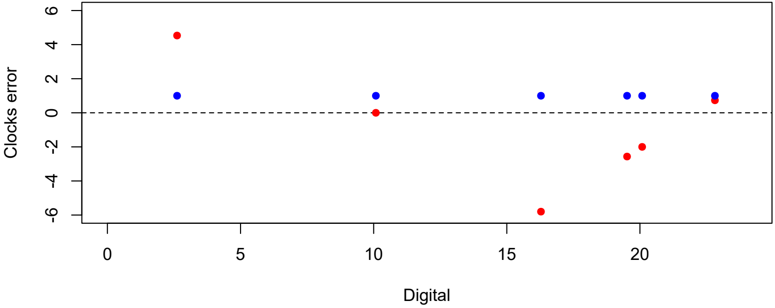

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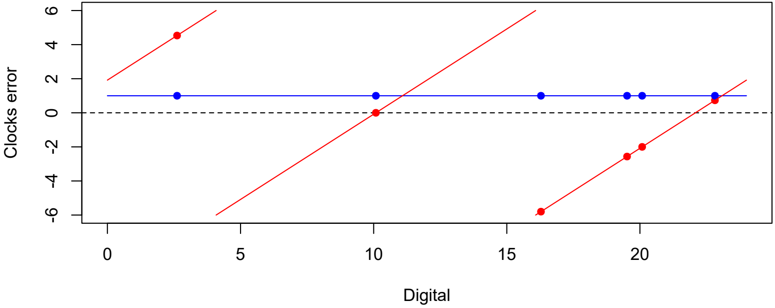

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The decimal clock has

$$E(\delta_2) = 0$$

and

$$\text{var}(\delta_2) = 12,$$

with

$$\delta_2 \sim \text{U}_{[-6,6]}.$$

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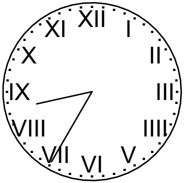

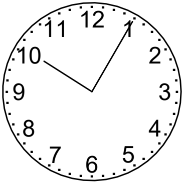

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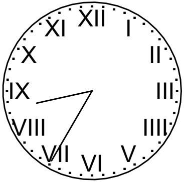
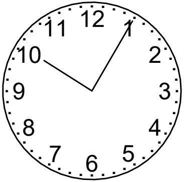
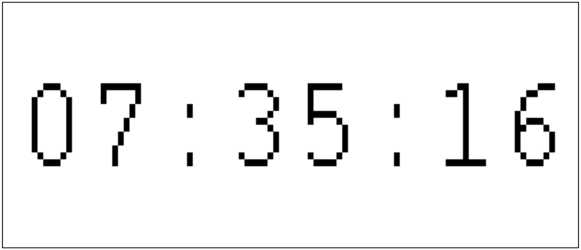
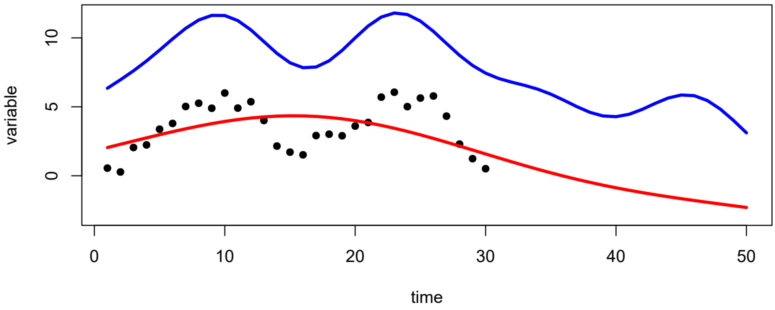






Discrepancy example



Modelling discrepancy

We know

$$x_{i,t} = y_t + \delta_{i,t}.$$

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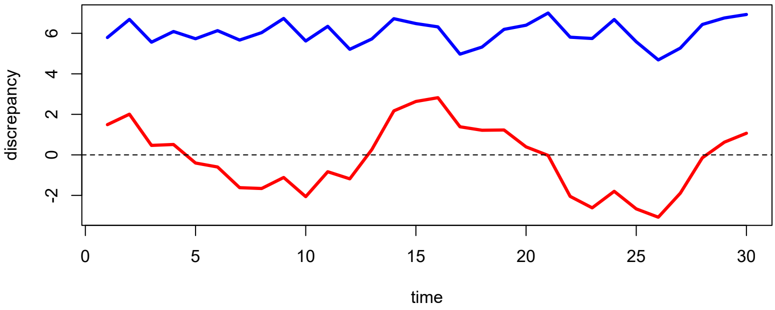

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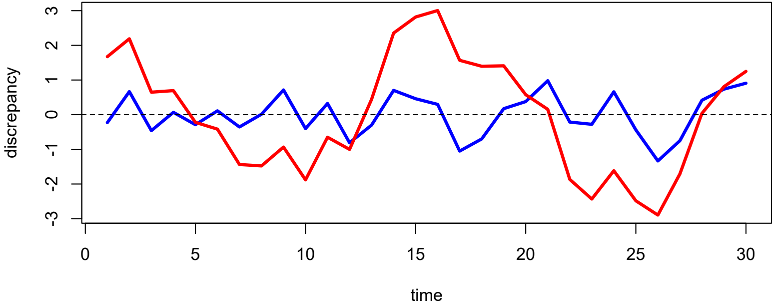

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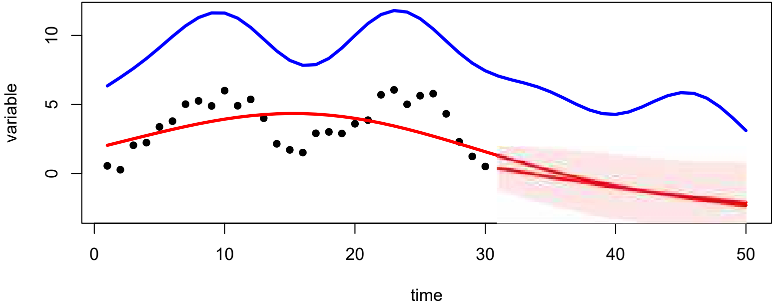
$$\zeta_{i,t} = \rho_i \zeta_{i,t-1} + \epsilon_{i,t}$$

and

$$\epsilon_{i,t} \sim N(0, \sigma_i^2).$$

Modelling discrepancy



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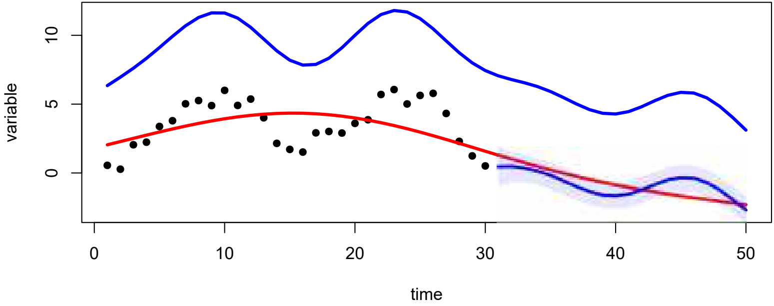

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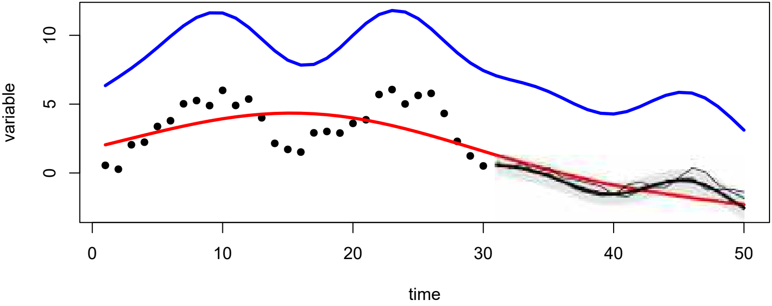

Modelling discrepancy

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Ensemble modelling can be mixed effects modelling

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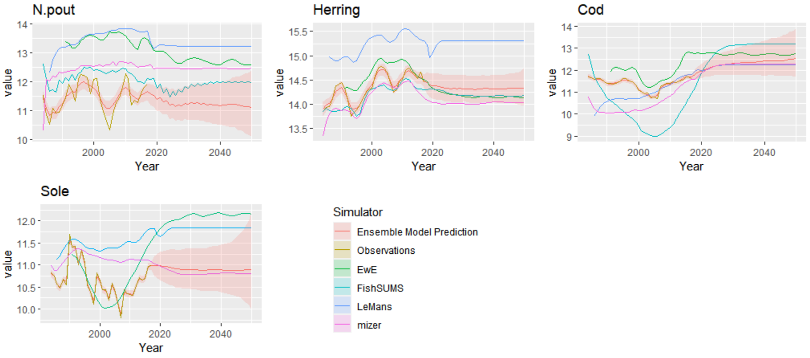
- These incorrect assuptions can lead to increase uncertainty with more knowledge
- Examining discrepancies can utilise information
- Ensemble modelling is a regression problem e.g. random effects model

EcoEnsemble



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- Spence et al. (2021) Sustainable fishing can lead to improvements in marine ecosystem status: an ensemble-model forecast of the North Sea ecosystem. *Mar Ecol Prog Ser* 680:207-221.
- Spence M. A, Martindale J. A & Thomson M. J (2022) EcoEnsemble: A General Framework for Combining Ecosystem Models. *https://CRAN.R-project.org/package=EcoEnsemble*

Subjective beliefs - example prePam

The beliefs after Dwight were

$$p(y|x_{\text{Kelly}}, \ldots, x_{\text{Dwight}}) = \begin{cases} \frac{1}{4} & \text{if } y = 9, 10, 11 \text{ or } 12 \\ 0 & \text{otherwise.} \end{cases}$$

with Pam being

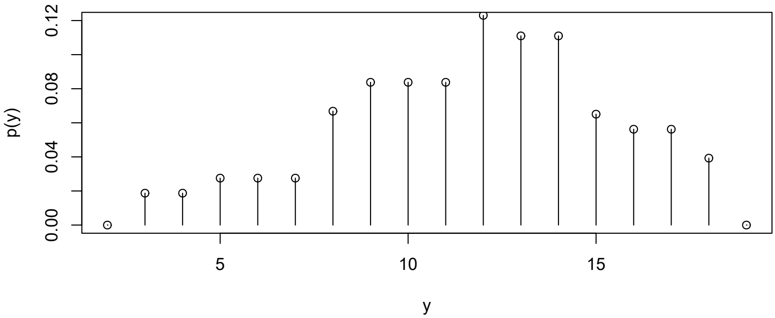
$$\begin{aligned} p(y|x_{\text{Pam}}) &= \frac{p(y)p(x_{\text{Pam}}|y)}{\sum_{z=0}^{100} p(z)p(x_{\text{Pam}}|z)} \\ &= \begin{cases} \frac{1}{7} & \text{if } y = 12, 13, 14, 15, 16, 17 \text{ or } 18 \\ 0 & \text{otherwise.} \end{cases} \\ &= p(x_{Pam}|y) \end{aligned}$$

Subjective beliefs - example including Pam

Adding Pam's information, we get

$$
\begin{aligned}
p(y|x_{\text{Kelly}},\ldots,x_{\text{Dwight}},x_{Pam}) &= \frac{p(y|x_{\text{Kelly}},\ldots,x_{\text{Dwight}})p(x_{Pam}|y)}{\sum_{z=0}^{100} p(z|x_{\text{Kelly}},\ldots,x_{\text{Dwight}})p(x_{\text{Pam}}|z)} \\
&= \begin{cases} 1 & \text{if } y = 12 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Weighted model – example weighted



Adding to our knowledge

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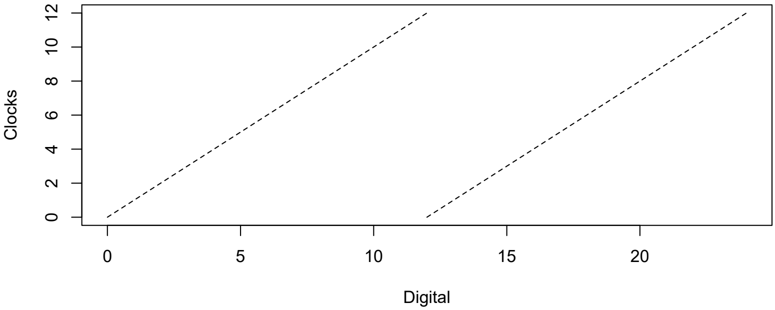
and so

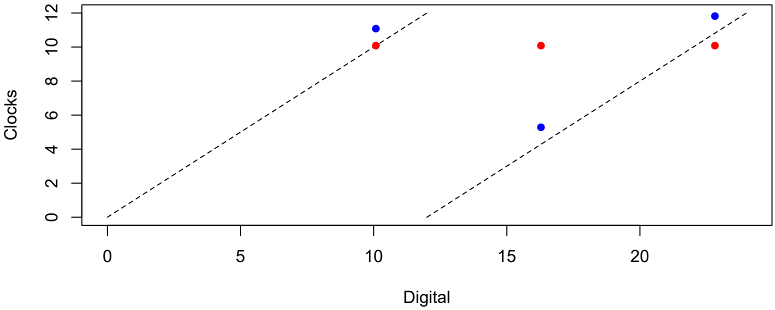
$$p(y|x_1, \ldots, x_{m+1}) = p(y|x_1, \ldots, x_m).$$

We learnt nothing.

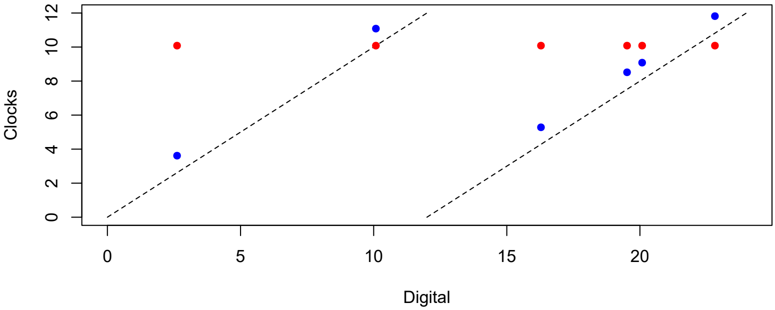
What time is it?



What time is it?



What time is it?



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for this document.